Differentialgleichung

Beispiel 1: Zucker wird in H_2O gelöst. Lösungsgeschwindigkeit v zu dem noch nicht gelöstem Zuckeranteil.

$z = z(t)$ = der in Zeit t gelöste Zuckeranteil

wir erhalten die Dgl.

$$\frac{dz}{dt} = k\,(100 - z) \qquad k = \text{Proportionalitätsfaktor}$$

Lösung der Dgl durch Trennung der Variablen

$$\frac{dz}{dt} = k\,(100 - z)$$

$$\frac{1}{k}\,\frac{dz}{(100-z)} = dt$$

$$\frac{1}{k}\int \frac{dz}{100-z} = \int dt$$

$$-\frac{1}{k}\ln|100-z| + C_1 = t + C_2$$

$$\ln|100-z|^{-\frac{1}{k}} = t + C_3 \qquad | \; C_3 = C_2 - C_1$$

$$e^{\ln|100-z|^{-\frac{1}{k}}} = e^{t + C_3}$$

$$e^{\ln|100-z|} = \left(e^{t + C_3}\right)^{-k}$$

$$(100-z)^{-\frac{1}{k}} = e^{-tk + C_3(-k)}$$

$$100 - z = e^{-tk} \cdot e^{-C_3 k}$$

$$100 - z = e^{-tk} \cdot C_4 \qquad | \; C_4 = e^{-C_3 k}$$

$$z = 100 - C_4 e^{-tk}$$

$$\Rightarrow \quad z(t) = 100 - C_4 \cdot e^{-tk}$$

wenn wir davon ausgehen daß nach 0 s auch 0 g Zucker gelöst sind, dann können wir eine Anfangsbedingung aufstellen und C_4 berechnen:

$$z(t=0) = 0 = 100 - C_4 \cdot e^0 = 100 - C_4 \cdot 1 \quad \Rightarrow \quad C_4 = 100$$

wir erhalten die partikuläre Lösung der Dgl.:

$$z_p(t) = 100 - 100 \cdot e^{-tk}$$

2) **Beispiel 2 Freier Fall**

$$F_G = m \cdot g$$
$$F_w = c \cdot v^2$$

$$F = m \cdot a \quad , \quad a = \frac{dv}{dt} \quad \Rightarrow \quad F = m \frac{dv}{dt}$$

$$\Rightarrow \quad m \cdot g - c \cdot v^2 = m \frac{dv}{dt}$$

$$\frac{dv}{dt} = g - \frac{c \cdot v^2}{m}$$

$$\frac{dv}{g - \frac{c}{m} v^2} = dt$$

$$\int \frac{dv}{g - \frac{c}{m} v^2} = \int dt = t$$

Nebenrechnung

$$\int \frac{dv}{g - \frac{c}{m} v^2} = \frac{1}{g} \int \frac{dv}{1 - \frac{c}{mg} v^2}$$

Subst. $z = \sqrt{\frac{c}{mg}}\, v$

$$\frac{dz}{dv} = \sqrt{\frac{c}{mg}} \quad \rightsquigarrow \quad dv = \frac{dz}{\sqrt{\frac{c}{mg}}} = \sqrt{\frac{mg}{c}}$$

$$\Rightarrow \quad = \sqrt{\frac{m}{c \cdot g}} \int \frac{dz}{1 - z^2} = \sqrt{\frac{m}{c \cdot g}} \cdot \operatorname{arctanh} z + K$$

Einsetzen in Dgl

$$t = \sqrt{\frac{m}{c \cdot g}} \cdot \operatorname{arctanh}\left(\sqrt{\frac{c}{m \cdot g}}\right) + K$$

Setzen wir als Anfangsbedingung fest $v(t) = 0$ so folgt $K = 0$

$$\Rightarrow \quad \operatorname{arctanh}\left(\sqrt{\frac{c}{m \cdot g}} \cdot v\right) = \sqrt{\frac{c \cdot g}{m}}\, t$$

$$\Rightarrow \quad \sqrt{\frac{c}{m \cdot g}}\, v = \tanh\left(\sqrt{\frac{c}{m \cdot g}}\, t\right)$$

$$v = \sqrt{\frac{m \cdot g}{c}} \cdot \tanh\left(\sqrt{\frac{c}{m \cdot g}} \cdot t\right)$$

Beispiel 3 Substitutionsmethode

$$y' = (x+y-1)^2 \quad \text{Anfangsbedingung} \quad y(x=0)=1 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

Substitution: $z = x+y-1$

Auflösen nach y: $y = z - x + 1$

$$y' = \frac{dy}{dx} = \frac{dz}{dx} - 1$$

einsetzen in Dgl.

$$\frac{dz}{dx} - 1 = z^2$$

$$\frac{dz}{dx} = z^2 + 1$$

$$\int \frac{dz}{z^2+1} = \int dx$$

$$\arctan z = x + c$$

$$z = \tan(x+c)$$

$$x+y-1 = \tan(x+c)$$

$$y = 1 - x + \tan(x+c) \quad \text{allgemeine Lösung}$$

Auswerten der Anfangsbed. $y(x=0)=1$

$$1 = 1 + \tan(c) \Rightarrow \tan c = 0 \Rightarrow c = 0$$

$$y = 1 - x + \tan x$$

Inhomogene Gleichung

$$y' \cdot \cos x + y \sin x = \tan x$$

Anfangsbedingung $\quad y(x=0) = \frac{\pi}{2}$

$$y' \cdot \cos x + y \sin x = \tan x \qquad | : \cos x$$

$$y' + y \frac{\sin x}{\cos x} = \underbrace{\frac{\tan x}{\cos x}}_{\text{Störfunktion}}$$

- **1. Schritt** Lösung der homogenen DGL.

$$y' + y \cdot \frac{\sin x}{\cos x} = 0$$

$$y' = \frac{dy}{dx} = -y \frac{\sin x}{\cos x}$$

Trennung der Variablen

$$\frac{dy}{y} = -\frac{\sin x}{\cos x}\, dx$$

$$\int \frac{dy}{y} = \int \frac{-\sin x}{\cos x}\, dx \qquad \text{Typ 3, Subst. } t = \cos x$$

$$\ln|y| = \ln|\cos x| + C_1$$

$$\ln|y| = \ln|\cos x| + \ln C_2 \qquad | \ C_1 = \ln C_2$$

$$|y| = e^{\ln|\cos x| + \ln C_2}$$

$$|y| = e^{\ln|\cos x|} \cdot e^{\ln C_2}$$

$$|y| = \cos x \cdot C_2$$

$$y = C_3 \cos x \qquad | \ C_3 = \pm C_2$$

- **2. Schritt** Variation der Konstanten

Ansatz $\quad y_p = C(x) \cdot \cos x$

$$y_p' = C'(x) \cdot \cos x - C(x) \cdot \sin x$$

Einsetzen in DGL

$$C'(t)\cdot\cos x - C(t)\cdot\sin x + C(t)\cdot\cos x\cdot\frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x}$$

$$C'(t)\cdot\cos x - C(t)\cdot\sin x + C(t)\cdot\sin x = \frac{\sin x}{\cos^2 x}$$

$$C'(t)\cdot\cos(t) = \frac{\sin x}{\cos^2 x}$$

$$C'(t) = \frac{\sin x}{\cos^3 x}$$

$$C(x) = \int\frac{\sin x}{\cos^3 x}\,dt$$ Substitution $\cos x = z \Rightarrow \frac{dz}{dt} = -\sin x$

$$dx = \frac{dz}{-\sin x}$$

$$\Rightarrow \int\frac{\sin x}{z^3}\,\frac{dz}{-\sin x} = -\int\frac{dz}{z^3} = -\int z^{-3}\,dz$$

$$= +\frac{1}{2}z^{-2} = \frac{1}{2\cos^2 x}$$ hier keine integrations Konstante

$\Rightarrow C(x) = \frac{1}{2\cos^2 x}$ einsetzen in den Ansatz $y_p = C(t)\cdot\cos x$ ergibt

$$y_p = \frac{1}{2\cos^2 x}\cdot\cos x = \frac{1}{2\cos x}$$

<u>3. Schritt</u> Zusammensetzen der allgemeinen Lösung

$$y_{allg} = y_h + y_p$$

$$y_{allg} = C_3\cos x + \frac{1}{2\cos x}$$

<u>4. Schritt</u> Auswerten der Anfangsbedingung

$$y(x=0) = \frac{5}{2} = C_3\cos(0) + \frac{1}{2\cos(0)} \Rightarrow$$ 5

$$\frac{5}{2} = C_3 + \frac{1}{2} \Rightarrow C_3 = 2$$

wir erhalten die spezielle Lösung

$$y_{spez} = 2\cos(x) + \frac{1}{2\cos(x)}$$

8

Bestimmung der konstanten C: $(y'(1) = \sqrt{3})$

$$y_{allg} = C \cdot (4-x^2)^{-\frac{1}{2}} - 1$$

$$y'_{allg} = -\frac{1}{2} C \cdot (4-x^2)^{-\frac{3}{2}} \cdot (-2x) = Cx(4-x^2)^{-\frac{3}{2}} = \frac{Cx}{\sqrt{(4-x^2)} \cdot (4-x^2)}$$

$$y'_{allg}(1) = \sqrt{3} \Rightarrow \quad \sqrt{3} = \frac{C}{\sqrt{4-1} \cdot 3} = \frac{C}{\sqrt{3} \cdot 3}$$

$$\sqrt{3} = \frac{C}{\sqrt{3} \cdot 3}$$

$$\sqrt{3} \cdot \sqrt{3} \cdot 3 = C = 9$$

$$\Rightarrow \quad y = \frac{9}{\sqrt{4-x^2}} - 1$$

11.02.04
Aufgabe 4

$$y'(1+x^2) + xy - \frac{1}{x^2} = 0$$

$$y'(1+x^2) + xy = \frac{1}{x^2}$$

1. homogene DgL

$$y'(1+x^2) + xy = 0 \quad !$$

umstellen nach $y' + \frac{xy}{1+x^2} = 0$

hier wg. Sonderfall nicht nötig

$$y'(1+x^2) = -xy$$

$$\frac{dy}{dx} = -\frac{xy}{1+x^2}$$

$$\frac{dy}{y} = -\frac{x\,dx}{1+x^2}$$

$$\int \frac{dy}{y} = -\int \frac{x\,dx}{1+x^2}$$

$$\ln|y| = -\frac{1}{2} \int \frac{2x}{1+x^2} \qquad (\text{Typ } 3)$$

$$\ln|y| = -\frac{1}{2} \ln|1+x^2| + C_1$$

$$|y| = e^{-\frac{1}{2}\ln|1+x^2| + C_1}$$

$$|y| = e^{\ln|1+x^2|^{-\frac{1}{2}}} \cdot e^{C_1}$$

$$y = |1+x^2|^{-\frac{1}{2}} \cdot C_2 \qquad | \; C_2 = \ln C_1$$

Einsetzen in die DgL

$$\cancel{\frac{C(x)\cdot x}{\sqrt{4-x^2}\cdot(4-x^2)}} + \frac{C'(x)}{\sqrt{4-x^2}} - \cancel{\frac{x\cdot C(x)}{\sqrt{4-x^2}\cdot(4-x^2)}} = \frac{x}{4-x^2}$$

$$\frac{C'(x)}{\sqrt{4-x^2}} = \frac{x}{4-x^2} \curvearrowright C'(x) = \frac{x}{4-x^2}\cdot\sqrt{4-x^2}$$

$$C'(x) = \frac{x}{\sqrt{4-x^2}}$$

$$C(x) = \int \frac{x}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4\left(1-\frac{x^2}{4}\right)}}\,dx$$

$$C(x) = \int \frac{x}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \int \frac{\frac{x}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

Substitution $\frac{x}{2} = \sin t$

$$x = 2\sin(t)$$

$$\frac{dx}{dt} = 2\cos(t) \curvearrowright dx = 2\cos(t)\,dt$$

$$c(x) = \int \frac{\sin(t)}{\sqrt{1-\sin^2(t)}}\cdot 2\cos(t)\,dt = 2\int \sin t\,dt$$

$$c(x) = -2\cos t \qquad \Big|\ \cos t = \sqrt{1-\sin^2 t}$$

$$c(x) = -2\sqrt{1-\sin^2 t}$$

Rücksubstitution

$$c(x) = -2\sqrt{1-\left(\frac{x}{2}\right)^2} = -\sqrt{4\cdot 4\left(\frac{x}{2}\right)^2} = -\sqrt{4\cdot x^2}$$

$$Y_p = \frac{c(x)}{\sqrt{4-x^2}} = \frac{-\sqrt{4-x^2}}{\sqrt{4-x^2}} = -1$$

Allgemeine Lösung:

$$Y_{allg} = Y_h + Y_p = \frac{C}{\sqrt{4-x^2}} - 1$$

Bestimmung der Konstanten C: $(y'(1)=\sqrt{3})$

$$y_{allg} = C\cdot(4-x^2)^{-\frac{1}{2}} - 1$$

$$y'_{allg} = -\frac{1}{2}C\cdot(4-x^2)^{-\frac{3}{2}}\cdot(-2x) = Cx(4-x^2)^{-\frac{3}{2}} = \frac{Cx}{\sqrt{(4-x^2)}\cdot(4-x^2)}$$

$$y'_{allg}(1) = \sqrt{3} \Rightarrow \quad \sqrt{3} = \frac{C}{\sqrt{4-1}\cdot 3} = \frac{C}{\sqrt{3}\cdot 3}$$

$$\sqrt{3} = \frac{C}{\sqrt{3}\cdot 3}$$

$$\sqrt{3}\cdot\sqrt{3}\cdot 3 = C = 9$$

$$\Rightarrow \quad y = \frac{9}{\sqrt{4-x^2}} - 1$$

11.02.04
Aufgabe 4

$$y'(1+x^2) + xy - \frac{1}{x^2} = 0$$

$$y'(1+x^2) + xy = \frac{1}{x^2}$$

1. homogene DgL

$$y'(1+x^2) + xy = 0$$

$$y'(1+x^2) = -xy$$

$$\frac{dy}{dx} = -\frac{xy}{1+x^2}$$

$$\frac{dy}{y} = -\frac{x\,dx}{1+x^2}$$

$$\int\frac{dy}{y} = -\int\frac{x\,dx}{1+x^2}$$

$$\ln|y| = -\frac{1}{2}\int\frac{2x}{1+x^2} \qquad (\text{Typ } 3)$$

$$\ln|y| = -\frac{1}{2}\ln|1+x^2| + C_1$$

$$|y| = e^{-\frac{1}{2}\ln|1+x^2| + C_1}$$

$$|y| = e^{\ln|1+x^2|^{-\frac{1}{2}}}\cdot e^{C_1}$$

$$y = |1+x^2|^{-\frac{1}{2}}\cdot C_2 \qquad | \; C_2 = \ln C_1$$

$$y_h = \frac{-2}{\sqrt{1+x^2}}$$ $y=0$ ist ebenfalls eine Lösung

2. Variation der Konstanten

$$y_p = \frac{c(x)}{\sqrt{1+x^2}} = c(x) \cdot (1+x^2)^{-\frac{1}{2}}$$

$$y_p' = c'(x) \cdot (1+x^2)^{-\frac{1}{2}} + c(x) \cdot -\frac{1}{2}(1+x^2)^{-\frac{3}{2}} \cdot 2x$$

$$y_p' = \frac{c'(x)}{\sqrt{1+x^2}} - \frac{c(x) \cdot x}{\sqrt{1+x^2}\,(1+x^2)}$$

3. einsetzen in die DgL

$$\left(\frac{c'(x)}{\sqrt{1+x^2}} - \frac{c(x) \cdot x}{\sqrt{1+x^2}\,(1+x^2)}\right) \cdot (1+x^2) + x\left(\frac{c(x)}{\sqrt{1+x^2}}\right) = \frac{1}{x^2}$$

$$c'(x) \cdot (1+x^2)^{-\frac{1}{2}} \cdot (1+x^2)^{\frac{2}{2}} - c(x) \cdot x (1+x^2)^{-\frac{3}{2}} \cdot (1+x^2)^{\frac{2}{2}} + c(x) \cdot x \cdot (1+x^2)^{-\frac{1}{2}} = \frac{1}{x^2}$$

$$c'(x) \cdot \sqrt{1+x^2} = \frac{1}{x^2}$$

$$c'(x) = \frac{1}{x^2 \cdot \sqrt{1+x^2}}$$

$$c(x) = \int \frac{1}{x^2 \cdot \sqrt{1+x^2}}\,dx$$

Substitution

$x = \sinh(t) \rightarrow t = \operatorname{arcsinh}(t)$

$\frac{dx}{dt} = \cosh(t) \rightarrow dx = \cosh(t)\,dt$

$$c(x) = \int \frac{1 \cosh(t)}{\sinh^2(t) \cdot \sqrt{1+\sinh^2(t)}}\,dt$$

$$c(x) = \int \frac{1}{\sin^2(t)}\,dt = -\coth(t) = -\frac{\cosh(t)}{\sinh(t)} = -\frac{\sqrt{1+\sinh^2(t)}}{\sinh(t)}$$

Rücksubst.: $c(x) = -\frac{\sqrt{1+x^2}}{x}$

$$y_{spez.} = \frac{c(x)}{\sqrt{1+x^2}} = -\frac{\sqrt{1+x^2}}{x\sqrt{1+x^2}} = -\frac{1}{x}$$

$$y_{allg} = y_h + y_s = \frac{c}{\sqrt{1+x^2}} - \frac{1}{x} \qquad c \in \mathbb{R}$$

12.02.03

A 4) $y' \cdot \cos(x) - y \cdot \sin(x) = \sin(2x)$

$P_0(\pi; -1)$ ist Anfangsbed.

1) homogene DgL

$$y' \cdot \cos(x) - y \cdot \sin(x) = 0$$

$$\frac{dy}{dx} \cdot \cos(x) = y \cdot \sin(x)$$

$$\frac{dy}{y} = \frac{\sin x}{\cos x}\, dx$$

$$\int \frac{dy}{y} = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx \quad (\text{Typ } 3)$$

$$\ln|y| = -\ln|\cos x| + C_1$$

$$|y| = e^{\ln|\cos x|^{-1} + c_1}$$

$$|y| = e^{\ln|\cos x|^{-1}} \cdot e^{\ln c_1}$$

$$|y| = \cos^{-1}(x) \cdot c_2 \qquad | \; c_2 = + \ln c_1$$

$$y = \cos^{-1}(x) \cdot C \qquad | \; C = \pm c_2$$

$$y_h = \frac{C}{\cos(x)}$$

2. Variation der Konstanten

$$y_p = \frac{C(x)}{\cos(x)} = C(x) \cdot \frac{1}{\cos(x)} = C(x) \cdot \cos^{-1}(x)$$

$$y'_p = C'(x) \cdot \cos^{-1}(x) + C(x) \cdot -\sin(x) \cdot -1 \cos^{-2}(x)$$

$$y'_p = \frac{C'(x)}{\cos(x)} + \frac{C(x) \cdot \sin x}{\cos^2(x)}$$

3. Einsetzen in DgL

$$\left(\frac{C'(x)}{\cos(x)} + \frac{C(x) \cdot \sin(x)}{\cos^2(x)}\right) \cdot \cos(x) - \frac{C(x)}{\cos(x)} \cdot \sin(x) = \sin(2x)$$

$$C'(x) + \frac{C(x) \cdot \sin(x)}{\cos(x)} - \frac{C(x)}{\cos(x)} \cdot \sin(x) = \sin(2x)$$

$c'(x) = \sin(2x)$

$c(x) = \int \sin(2x)\,dx \qquad t = 2x \Rightarrow \frac{dt}{dx} = 2 \quad \curvearrowright \quad dx = \frac{dt}{2}$

$c(x) = \int \sin t \ \frac{dt}{2}$

$c(x) = -\frac{\cos(2x)}{2}$

$y_p = -\frac{\cos(2x)}{2\cos(x)}$

$y_{allg} = y_h + y_p = \frac{c}{\cos(x)} - \frac{\cos(2x)}{2\cos(x)}$

- 4. Auswerten der Anfangsbedingung

$P_0(\pi ; -1) \Rightarrow y(x=\pi) = -1$

$-1 = \frac{c}{\cos(\pi)} - \frac{\cos(2\pi)}{2 \cdot \cos(\pi)}$

$-1 = -c + \frac{1}{2}$

$-\frac{3}{2} = -c \Rightarrow c = \frac{3}{2}$

- $y_{spez} = \frac{3}{2\cos(x)} - \frac{\cos(2x)}{2 \cdot \cos(x)}$

06.02.02

A4

$$y' - \frac{xy}{4-x^2} + \frac{1}{4-x^2} = 0$$

$$\Rightarrow \quad y' - \frac{xy}{4-x^2} = -\frac{1}{4-x^2}$$

1. homogene DGL

$$y' - \frac{xy}{4-x^2} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{xy}{4-x^2} \quad \Rightarrow \quad \frac{dy}{y} = -\frac{x}{4-x^2}\,dx$$

$$\int \frac{dy}{y} = \int \frac{x}{4-x^2}\,dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\frac{1}{2}\int \frac{-2x}{4-x^2}\,dx \qquad (\text{Typ } 3)$$

$$\ln|y| = -\frac{1}{2}\ln|4-x^2| + c_1$$

$$|y| = e^{-\frac{1}{2}\ln|4-x^2|} \cdot e^{c_1}$$

$$|y| = e^{\ln|4-x^2|^{-\frac{1}{2}}} \cdot e^{c_1}$$

$$y = (4-x^2)^{-\frac{1}{2}} \cdot c_2 \qquad | \; c_2 = \pm \ln c_1$$

$$y_h = \frac{c_2}{\sqrt{4-x^2}}$$

2 Variation der Konstanten

$$y_p = \frac{c(x)}{\sqrt{4-x^2}} = c(x) \cdot (4-x^2)^{-\frac{1}{2}}$$

$$y_p' = c'(x) \cdot (4-x^2)^{-\frac{1}{2}} + c(x) \cdot -\frac{1}{2}(4-x^2)^{-\frac{3}{2}} \cdot -2x$$

$$y_p' = c'(x) \cdot (4-x^2)^{-\frac{1}{2}} + c(x) \cdot (4-x^2)^{-\frac{3}{2}}$$

3. Einsetzen in DGL

$$c'(x) \cdot (4-x^2)^{-\frac{1}{2}} + x\,c(x) \cdot (4-x^2)^{-\frac{3}{2}} - \frac{x \cdot c(x) \cdot (4-x^2)^{-\frac{1}{2}}}{4-x^2} = \frac{-1}{4-x^2}$$

$$c'(x) \cdot (4-x^2)^{-\frac{1}{2}} + x\,c(x) \cdot (4-x^2)^{-\frac{3}{2}} - x \cdot c(x) \cdot (4-x^2)^{-\frac{1}{2}} \cdot (4-x^2)^{-\frac{3}{2}} = -\frac{1}{4-x^2}$$

$$c'(x) = -\frac{1}{4-x^2} \cdot (4-x^2)^{-\frac{1}{2}}$$

$$c(x) = -\int \frac{1}{4-x^2} \cdot (4-x^2)^{\frac{1}{2}}\, dx = -\int (4-x^2)^{-\frac{2}{2}} \cdot (4-x^2)^{\frac{1}{2}}\, dx$$

$$c(x) = -\int \frac{1}{\sqrt{4-x^2}}\, dx = -\int \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}}\, dx = -\int \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$c(x) = -\frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\, dx$$

Subst. $t = \frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2} \rightarrow dx = \frac{dt}{\frac{1}{2}} = 2\,dt$

$$c(x) = -\int \frac{1}{\sqrt{1-t^2}}\, dt = -\arcsin(t) = -\arcsin\left(\frac{x}{2}\right)$$ (Grundintegral)

$$y_p = \frac{c(x)}{\sqrt{4-x^2}} = \frac{-\arcsin\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}$$

$$y_{allg} = y_h + y_p = \frac{c_2}{\sqrt{4-x^2}} - \frac{\arcsin\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}$$

28.09.04
A 4

$$\frac{y'}{\sin(x)} + y = 1$$! umstellen!

$$y' + y \cdot \sin(x) = \sin(x)$$

1. homogene DgL

$$y' + y\sin(x) = 0$$

$$\frac{dy}{dx} + y\sin(x) = 0$$

$$\frac{dy}{dx} = -y\sin x$$

$$\frac{dy}{y} = -\sin x\, dx$$

$$\int \frac{dy}{y} = -\int \sin(x)\, dx$$

$$\ln|y| = -(-\cos x) + c_1$$

$$|y| = e^{\cos x + c_1} = e^{\cos x} \cdot e^{c_1}$$

$$|y| = e^{\cos x} \cdot c_2 \qquad | c_2 = \ln c_1$$

$$y = e^{\cos x} \cdot c \qquad | c = \pm c_2$$

2. Variation der Konstanten

$$y_p = c(x) \cdot e^{\cos x}$$

$$y_p' = c'(x) \cdot e^{\cos x} + c(x) \cdot (-\sin x) \cdot e^{\cos x}$$

3. Einsetzen in DGL

$$\frac{c'(x) \cdot e^{\cos x}}{\sin x} - \frac{c(x) \cdot \sin(x) \cdot e^{\cos x}}{\sin x} + e^{\cos(x)} \cdot c(x) = 1$$

$$\frac{c'(x) \cdot e^{\cos x}}{\sin x} = 1$$

$$c'(x) = \frac{\sin x}{e^{\cos x}}$$

$$c(x) = \int \sin x \cdot e^{-\cos x}\, dx \qquad \text{subst} \quad u = -\cos x$$

$$\frac{du}{dx} = \sin x \quad \wedge \quad dx = \frac{du}{\sin x}$$

$$c(x) = \int \sin x \cdot e^{u} \cdot \frac{du}{\sin x}$$

$$c(x) = \int e^{u}\, du = e^{u} = e^{-\cos x}$$

$$y_p = e^{-\cos x} \cdot e^{\cos x} = \frac{e^{\cos x}}{e^{\cos x}} = 1$$

$$y_{allg} = y_h + y_p = e^{\cos x} \cdot C + 1$$

4 Auswerten der Anfangsbed.

$$y\left(x = \frac{\pi}{2}\right) = 2$$

$$\Rightarrow \quad 2 = e^{\cos\left(\frac{\pi}{2}\right)} \cdot C + 1 \quad \Rightarrow \quad 2 = e^{0} \cdot C + 1$$

$$1 = C \cdot e^{0}$$

$$C = 1$$

18.09.01
A4

$$y' + \frac{y}{x} = \cos(x)$$

Anfangsbed. $y(x = \frac{\pi}{2}) = 2$

1. Lösen der homogenen Dgl

$$y' + \frac{y}{x} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x} \quad \sim \quad \frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln|y| = -\ln|x| + C_1$$

$$|y| = e^{-\ln|x|} \cdot e^{C_1}$$

$$|y| = e^{\ln|x|^{-1}} \cdot C_2 \qquad | C_2 = \ln C_1$$

$$y_h = \frac{1}{x} \cdot C_3 \qquad | C_3 = \pm C_2$$

$$y_h = \frac{C_3}{x}$$

2. Variation der Konstanten

$$y_p = \frac{C(x)}{x} = C(x) \cdot x^{-1}$$

$$y_p' = C'(x) \cdot x^{-1} + C(x) \cdot x^{-2} \cdot (-1) = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$$

3. Einsetzen in Dgl

$$\frac{C'(x)}{x} - \frac{C(x)}{x^2} + \frac{C(x)}{x \cdot x} = \cos(x)$$

$$\frac{C'(x)}{x} = \cos(x)$$

$$C'(x) = x \cdot \cos(x)$$

$$C(x) = \int x \cdot \cos(x) \qquad \begin{matrix} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{matrix} \Rightarrow u \cdot v - \int v \cdot u' \, dx$$

$$\Rightarrow C(x) = x \cdot \sin x - \int \sin x \cdot 1 \, dx$$

$$C(x) = x \cdot \sin x - (-\cos(x))$$

$$y_p = \frac{C(x)}{x} = \frac{x \cdot \sin x + \cos x}{x} = \sin x + \frac{\cos x}{x}$$

$$y_{allg} = y_h + y_p = \frac{C_3}{x} + \sin x + \frac{\cos x}{x}$$

4. Auswerten der Anfangsbed. $y(x = \frac{\pi}{2}) = 2$

$$\Rightarrow \quad 2 = \frac{C_3}{\frac{\pi}{2}} + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + \underbrace{\frac{\cos\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}}_{0}$$

$$2 = \frac{2\,C_3}{\pi} + 1$$

$$1 = \frac{2\,C_3}{\pi} \quad \Rightarrow \quad C_3 = \frac{\pi}{2}$$

$$y_{spez.} = \frac{\frac{\pi}{2} \cdot \cos x}{x} + \sin x$$

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$$\cos(x) \cdot y' = y \cdot \sin x + \cos^2 x$$

$$\cos(x) \cdot y' - \sin x \; y = \cos^2 x$$

$$y' - \frac{y \sin x}{\cos(x)} = \frac{\cos^2 x}{\cos x} = \cos(x)$$

1. hom. DGL.

$$y' - \frac{y \sin x}{\cos x} = 0$$

$$\frac{dy}{dx} = \frac{y \sin x}{\cos x}$$

$$\frac{dy}{y} = \frac{\sin x}{\cos x}\,dx$$

$$\int \frac{dy}{y} = \int \frac{\sin x}{\cos x}\,dx = -1 \int \frac{-\sin x}{\cos x}\,dx \qquad (\text{Typ } 4)$$

$$\ln|y| = -\ln|\cos x| + C_1$$

$$|y| = e^{-\ln|\cos x| + C_1}$$

$$|y| = e^{-\ln|\cos x|^{-1}} \cdot e^{c_1}$$

$$|y| = (\cos x)^{-1} \cdot c_2 \qquad | c_2 = \ln c_1$$

$$y = \frac{c_3}{\cos x} \qquad | c_3 = \pm c_2$$

2. Variation der konst.

$$y_p = \frac{c(x)}{\cos x} = c(x) \cdot \cos^{-1}(x)$$

$$y_p' = c'(x) \cdot \cos^{-1}(x) + c(x) \cdot (-1)\cos^{-2}(x) \cdot (-\sin(x))$$

$$y_p' = c'(x) \cdot \cos^{-1}(x) + c(x) \cdot \cos^{-2}(x) \cdot \sin(x)$$

3. Einsetzen in Dgl

$$\frac{c'(x)}{\cos(x)} + \frac{c(x)}{\cos^2(x)} \cdot \sin x - \frac{c(x)}{\cos(x)} \cdot \frac{\sin x}{\cos(x)} = \cos x$$

$$\frac{c'(x)}{\cos(x)} = \cos(x)$$

$$c'(x) = \cos^2(x)$$

$$c(x) = \int \cos^2(x) \qquad (\text{Typ } 4)$$

$$c(x) = \frac{1}{2}(x + \sin x \cdot \cos x)$$

$$y_p = \frac{c(x)}{\cos x} = \frac{1}{2} \frac{(x + \sin x \cdot \cos x)}{\cos(x)} = \frac{1}{2}\left(\frac{x}{\cos x} + \sin x\right)$$

$$y_{allg} = y_h + y_p = \frac{c_3}{\cos(x)} + \frac{1}{2}\left(\frac{x}{\cos x} + \sin x\right)$$

$$= \frac{1}{2} \frac{2c_3}{\cos(x)} + \frac{1}{2}\left(\frac{x}{\cos x} + \sin x\right)$$

$$= \frac{1}{2}\left(\frac{2c_3 + x}{\cos x} + \sin x\right)$$

$$= \frac{1}{2}\left(\frac{c_4 + x}{\cos x} + \sin x\right) \qquad | c_4 = 2 \cdot c_3$$

18 13.03.01
44

$$y' + 2y \cdot \tan(t) = 2\tan(x)$$

1 homogene Dgl.

$$y' + 2y \cdot \tan(t) = 0$$

$$\frac{dy}{dx} + 2y\frac{\sin x}{\cos x} = 0$$

$$\frac{dy}{dx} = -2y\frac{\sin x}{\cos x}$$

$$\frac{dy}{2y} = -\frac{\sin x}{\cos x}\,dx$$

$$\frac{1}{2}\int \frac{1}{y}\,dy = \int \frac{-\sin x}{\cos x}\,dx \quad (\text{Typ } 3)$$

$$\frac{1}{2}\ln|y| = \ln|\cos x| + C_1$$

$$\ln|y| = 2(\ln|\cos x| + C_1) = 2\ln|\cos x| + C_2 \qquad | C_2 = 2\cdot C_1$$

$$|y| = e^{2\ln|\cos x| + C_2} = e^{\ln|\cos x|^2} \cdot e^{C_2}$$

$$|y| = \cos^2 x \cdot C_3 \qquad | C_3 = \ln C_2$$

$$y_h = \cos^2 x \cdot C_4 \qquad | C_4 = \pm C_3$$

2. Variation der Konstanten

$$y_p = C(x) \cdot \cos^2 x$$

$$y_p' = C'(x) \cdot \cos^2 x - C(x) \cdot 2\cos x \cdot \sin x$$

einsetzen in Dgl

$$C'(x) \cdot \cos^2 x - C(x) \cdot 2\cos x \cdot \sin x + 2C(x) \cdot \cos^2 x \cdot \frac{\sin x}{\cos x} = 2\frac{\sin x}{\cos x}$$

$$C'(x) \cdot \cos^2 x = 2\frac{\sin x}{\cos x}$$

$$C'(x) = \frac{2\sin x}{\cos^3 x}$$

$$C(x) = 2\int \frac{\sin x}{\cos^3 x}\,dx$$

Substitution $t = \cos x \curvearrowright \frac{dt}{dx} = -\sin x \Leftrightarrow dx = \frac{dt}{-\sin x}$

$$\Rightarrow 2\int \frac{\sin x}{-\sin x \cdot t^3}\,dt = -2\int \frac{1}{t^3}\,dt = -2\int t^{-3}\,dt = -2\,\frac{1}{-2}\,t^{-2} = +t^{-2}$$

$$C(x) = \frac{1}{t^2} = \frac{1}{\cos^2 x}$$

$$y_p = \frac{\cos^2 x}{\cos^2 x} = 1$$

$$y_{allg} = y_h + y_p = \cos^2 x \cdot C_4 + 1$$

Auswerten der Anfangsbed.

$$y\left(\frac{\pi}{4}\right) = 3 = \cos^2\left(\frac{\pi}{4}\right)\cdot C_4 + 1$$

$$3 = \left(\frac{\sqrt{2}}{2}\right)^2 \cdot C_4 + 1$$

$$2 = \frac{2}{4}\cdot C_4 \Rightarrow C_4 = 4$$

$$y_{spez} = 4\cos^2(x) + 1$$

24.09.02
14

$$y' + 2xy - xe^{-x^2} = 0$$

Anfangsbed. $y(1) = e$

$$y' + 2xy = xe^{-x^2}$$

1. hom. DgL

$$\frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} = -2xy$$

$$\frac{dy}{y} = -2x\,dx$$

$$\int \frac{dy}{y} = \int -2x\,dx$$

$$\ln|y| = -2\,\frac{1}{2}x^2 + C_1$$

$$|y| = e^{-x^2 + C_1} = e^{-x^2}\cdot e^{C_1} = e^{-x^2}\cdot C_2 \qquad | C_2 = \ln C_1$$

$$y_h = e^{-x^2}\cdot C \qquad | C = \pm C_2$$

2. Variation der Konstanten

$$y_p = e^{-x^2} \cdot C(x)$$
$$y_p' = C'(x) \cdot e^{-x^2} - C(x) \cdot 2x\, e^{-x^2}$$

einsetzen in DGL

$$C'(x) \cdot e^{-x^2} - C(x) \cdot 2x\, e^{-x^2} + 2x\, e^{-x^2}\, C(x) = x\, e^{-x^2}$$

$$C'(x) \cdot e^{-x^2} = x\, e^{-x^2}$$

$$C'(x) = x$$
$$C(x) = \frac{1}{2} x^2$$

$$y_p = e^{-x^2} \cdot \frac{1}{2} x^2$$

3. Allgemeine Lösung

$$y_{allg} = y_h + y_p = e^{-x^2} \cdot C_3 + e^{-x^2} \cdot \frac{1}{2} x^2 = e^{-x^2}\left(C + \frac{1}{2} x^2\right)$$

4. Auswerten der Anfangsbed.

$$y(1) = e$$
$$e = e^{-1}\left(C + \frac{1}{2}\right)$$
$$\frac{e}{e^{-1}} = C + \frac{1}{2}$$
$$e^2 - \frac{1}{2} = C$$
$$y_{spez} = e^{-x^2}\left(e^2 - \frac{1}{2} + \frac{1}{2} x^2\right)$$